

# CAN WE MAKE A FINSLER METRIC COMPLETE BY A TRIVIAL PROJECTIVE CHANGE?

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**ABSTRACT.** A trivial projective change of a Finsler metric  $F$  is the Finsler metric  $F + df$ . I explain when it is possible to make a given Finsler metric both forward and backward complete by a trivial projective change.

The problem actually came from lorentz geometry and mathematical relativity: it was observed that it is possible to understand the light-line geodesics of a (normalized, standard) stationary 4-dimensional space-time as geodesics of a certain Finsler Randers metric on a 3-dimensional manifold. The trivial projective change of the Finsler metric corresponds to the choice of another 3-dimensional slice, and the existence of a trivial projective change that is forward and backward complete is equivalent to the global hyperbolicity of the space-time.

## 1. STATEMENT OF THE PROBLEM, MOTIVATION AND THE MAIN RESULT

Let  $(M, F)$  be a connected Finsler manifold and  $f : M \rightarrow \mathbb{R}$  be a function such that

$$(1.1) \quad F(x, v) + d_x f(v) > 0 \quad \text{for all } (x, v) \in TM \text{ with } v \neq 0.$$

By a *trivial projective change* we understand the Finsler metric  $F + df$ .

It is customary in Finsler geometry to require the Finsler metric to be *strongly convex*, that is the Hessian of the restriction of  $F^2$  to  $T_x M \setminus \{0\}$  is assumed to be positive definite for any  $x \in M$ . Our results do not require this assumption and are valid also for Finsler metrics that are not strictly convex. Let us note though that if the metric  $F$  is strictly convex then the trivial projective change  $F + df$  is also strictly convex.

The metric  $F + df$  has the same unparameterized geodesics as  $F$ . Indeed, a forward-geodesic connecting two points  $x, y \in M$  is an extremal of the forward-length functional

$$(1.2) \quad L_F^+(c) := \int_a^b F(c(t), \dot{c}(t)) dt$$

in the set of all smooth curves  $c : [a, b] \rightarrow M$  connecting  $x$  and  $y$ . Now, replacing  $F$  by  $F + df$  in (1.2), we obtain

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$$(1.3) \quad L_{F+df}^+(c) := \int_a^b [F(c(t), \dot{c}(t)) + d_{c(t)}f(\dot{c}(t))] dt = L_F^+(c) + f(y) - f(x).$$

We see that the difference  $L_{F+df}^+(c) - L_F^+(c)$  is the constant  $f(y) - f(x)$  so the extremals of the functional (1.2) are extremals of the functional (1.3) and vice versa.

Analogically, for the backward-length

$$L_F^- := \int_a^b F(c(t), -\dot{c}(t)) dt,$$

we obtain  $L_{F+df}^-(c) - L_F^-(c) = f(x) - f(y)$  implying the backward-geodesics of  $F$  and  $F + df$  coincide.

Note that, though the unparameterized geodesics of  $F$  and  $F + df$  coincide, the arc-length parameter of the geodesics and also the distance functions generated by the Finsler metrics do not coincide (unless  $f$  is constant.) More precisely, the forward and backward distance functions

$$\text{dist}_F^\pm(x, y) = \inf\{L_F^\pm(c) \mid c : [a, b] \rightarrow M \text{ with } c(a) = x, c(b) = y\}$$

and the corresponding distance functions for  $F + df$  are related by

$$(1.4) \quad \text{dist}_{(F+df)}^\pm(x, y) = \text{dist}_F^\pm(x, y) \pm (f(y) - f(x)).$$

The goal of this note is to answer the questions under what conditions one could make a Finsler metric simultaneously forward complete and backward complete by an appropriate trivial projective change. We will assume that all objects we consider in our paper are sufficiently smooth. The assumption that the metric  $F$  is smooth is very natural in view of our motivation (see §1.1 below). We will see that the restriction that the (searched) function  $f$  is smooth (which is also natural in view of the motivation) actually makes our proof more complicated: if we allow the Lipschitz functions (and in Remark 3 we explain why we may do it), the proof becomes shorter and does not require the Appendix where we proved that it is possible, for any  $\varepsilon_1, \varepsilon_2 > 0$ , to  $\varepsilon_1$ -approximate an 1-Lipschitz function by a  $1 + \varepsilon_2$ -Lipschitz function, where the Lipschitz property is understood with respect to the (nonsymmetric) distance function coming from the metric  $F$ .

**1.1. Motivation.** Our motivation to study this question came from the mathematical relativity and Lorentz differential geometry. Following [4, 5, 6, 7], see also references therein, we consider the (normalized, standard) stationary space-time  $(M^4 = \mathbb{R} \times S^3, G)$ . Here  $S$  is a 3-dimensional manifold. The condition that the space-time is normalized, standard, stationary means that, in any local coordinate system  $(t, x^1, x^2, x^3)$  where  $t$  is the coordinate on  $\mathbb{R}$  and  $x^1, x^2, x^3$  are local coordinates on a 3-manifold  $S$ , the metric  $G$  is

given by the formula

$$(1.5) \quad \begin{aligned} G &= -dt^2 + 2\omega_i dx^i dt + g_{ij} dx^i dx^j \\ &= -(dt - \omega_i dx^i)(dt - \omega_j dx^j) + (g_{ij} + \omega_i \omega_j) dx^i dx^j, \end{aligned}$$

where  $g = g(x)_{ij}$   $i, j = 1, \dots, 3$ , is a Riemannian metric on  $S$  and  $\omega = \omega(x)_i$  is a 1-form on  $S$ .

**Remark 1.** *Note that the above definition of the normalized, standard, stationary spacetime is not the usual one. Usually, one defines a standard stationary spacetime as the one which is causal (i.e. it does not admit closed causal curves) and which admits a complete time-like Killing vector field  $K$ . By [9], this is equivalent to the condition that  $M$  is isometric to a product  $\mathbb{R} \times S$ , where  $S$  is some (appropriate) space-like hypersurface, and  $K$ , in the coordinate system corresponding to this decomposition, is the vector field  $\frac{\partial}{\partial t}$ . Then, the metric can be written locally as in (1.5); it is easy to check that  $\omega_i$  and  $g_{ij}$  could be viewed as objects globally defined on  $S$  since for  $\xi \in TS$  we have  $\omega(\xi) = G(\frac{\partial}{\partial t}, \xi) = G(K, \xi)$  and  $g$  is simply the restriction of  $G$  to  $S$ .*

Next, on  $S$ , we consider the (Randers) Finsler metric

$$(1.6) \quad F(x, \dot{x}) = \sqrt{(g(x)_{ij} + \omega(x)_i \omega(x)_j) \dot{x}^i \dot{x}^j} + \omega(x)_i \dot{x}^i.$$

As it was observed and actively studied in [4, 5, 8, 10], this Finsler metric and the initial Lorentz metric  $G$  are closely related. In particular, for every light-like geodesic  $\gamma(\tau) = (t(\tau), x^1(\tau), x^2(\tau), x^3(\tau))$  of  $G$ , its “projection” to  $S$ , i.e., the curve  $\tau \mapsto (x^1(\tau), x^2(\tau), x^3(\tau))$  on  $S$  is a (probably, reparameterized) geodesic of the Finsler metric (1.6). Moreover, the slice

$$(1.7) \quad \{0\} \times S = \{(0, x) \mid x \in S\}$$

is a Cauchy hypersurface of  $(M, G)$  if and only if the metric  $F$  is forward and backward complete, see [4, Theorem 4.4].

Note that it is possible to take another decomposition of  $M$  in the product of  $\mathbb{R} \times S'$  such that the metric  $G$  written in the coordinates adapted to the new decomposition still has the form (1.5) with possibly different  $g$  and  $\omega$ .

Indeed, consider another local coordinate systems  $(t', x^1, x^2, x^3)$  such that  $t' = t + f(x^1, x^2, x^3)$  (and the coordinates  $x^1, x^2, x^3$  are the same). Physically, this choice of the coordinates corresponds to the choice of another space-like slice: by the “old” slice we understand the 3-dimensional submanifold (1.7), and by the new one we understand  $\{(f(x), x) \mid x \in S\}$ .

In the new coordinates  $(t', x^1, x^2, x^3)$ , in view of  $dt' = dt - df$ , the metric  $G$  reads

$$(1.8) \quad - \left( dt' - \underbrace{\left( \omega_i + \frac{\partial f}{\partial x^i} \right)}_{\omega'_i} dx^i \right) \left( dt' - \underbrace{\left( \omega_j + \frac{\partial f}{\partial x^j} \right)}_{\omega'_j} dx^j \right) + (g_{ij} + \omega_i \omega_j) dx^i dx^j.$$

We see that the Finsler metric (1.6) constructed by the metric (1.8) is related to the initial metric (1.6) constructed by (1.5) by the formula  $F' = F + df$ , i.e., is the trivial projective change of the metric (1.6). It is easy to check that the slice  $\{(f(x), x) \mid x \in S\}$  is space-like if and only if

$$\sqrt{(g_{ij} + \omega_i \omega_j) \dot{x}^i \dot{x}^j} + \omega_i \dot{x}^i + \frac{\partial f}{\partial x^i} \dot{x}^i \text{ is positive for all } \dot{x}^i \neq 0,$$

i.e., if and only if  $f$  satisfies the condition from the definition of the trivial projective change with respect to the Finsler metric (1.6).

Thus, the question we are study, i.e., the existence of a trivial projective change of a Finsler metric such that the result is forward and backward complete is, in the special case when the metric is the Randers metric coming from the Lorentz metric (1.5) by the formula (1.6), equivalent to the existence of the function  $f$  such that the corresponding slice  $\{(f(x), x) \mid x \in S\}$  is a Cauchy hypersurface. Note that if such Cauchy hypersurface exists then the space-time is globally hyperbolic, see [2, 3] for details, and global hyperbolicity is an important condition to be studied in any space-time.

It appears though that the special form of the metric  $F$  suggested by the motivation does not make (our version of) the answer simpler, so we give the answer for the general Finsler metrics. It seems that even in the well studied situation when the Finsler metric is a Randers one, i.e., in the situation suggested by relativity, our main result which is Theorem 1 below is new; cf. [4, Theorem 5.10]. Actually, [4, Theorem 5.10] and our main Theorem 1 restricted to the Randers metrics are very similar: the difference is that in [4, Theorem 5.10] one (essentially) assumes that the function  $D^+ + D^-$  (see below) is proper for all choices of the point  $p$  as the initial point and we require this for one point  $p$  only.

**1.2. Main result.** We fix an arbitrary point  $p \in M$  and consider the functions  $D^+, D^- : M \rightarrow \mathbb{R}$  given by

$$D^\pm(x) := \text{dist}^\pm(p, x).$$

**Theorem 1.**  *$F$  can be made forward and backward complete by a trivial projective change if and only if the function  $D^+ + D^-$  is proper.*

Recall that a (continuous) function is *proper*, if the preimage of every compact set is compact or empty. Since the function  $D^+ + D^-$  is nonnegative and  $D^+(p) + D^-(p) = 0$ , the function  $D^+ + D^-$  is proper if and only if for every  $c \in \mathbb{R}_{\geq 0}$  the set

$$(1.9) \quad \{x \in M \mid D^+(x) + D^-(x) \leq c\}$$

is compact.

## 2. PROOF OF THEOREM 1.

First observe that if the function  $D^+ + D^-$  is proper then the function  $\alpha_1 D^+ + \alpha_2 D^-$  is proper for arbitrary positive numbers  $\alpha_i$ , and that if the

function  $\alpha_1 D^+ + \alpha_2 D^-$  is proper for some positive numbers  $\alpha_i$  then the function  $D^+ + D^-$  is proper. Indeed, the set

$$(2.1) \quad \{x \in M \mid \alpha_1 D^+(x) + \alpha_2 D^-(x) \leq c\}$$

is a (evidently, closed) subset of

$$\{x \in M \mid D^+(x) + D^-(x) \leq \frac{c}{\alpha_{\min}}\}, \quad \text{where } \alpha_{\min} = \min(\alpha_1, \alpha_2)$$

Then, if all the sets of the form (1.9) are compact, all sets of the form (2.1) are compact as well implying the function  $\alpha_1 D^+(x) + \alpha_2 D^-(x)$  is proper.

Now, if the function  $\alpha_1 D^+ + \alpha_2 D^-$  is proper, the set (1.9) is compact as a closed subset of

$$\{x \in M \mid \alpha_1 D^+ + \alpha_2 D^- \leq \alpha_{\max} c\}, \quad \text{where } \alpha_{\max} = \max(\alpha_1, \alpha_2),$$

implying  $D^+ + D^-$  is proper.

We will now show Theorem 1 in the direction “ $\implies$ ”: we show that if the function  $D^+ + D^-$  is not proper, then no projective change  $F + df$  is complete. Let  $R > 0$  be the number such that the set

$$B_R := \{x \in M \mid D^+(x) + D^-(x) \leq R\}$$

is not compact. For any  $x \in B_R$ , we have  $D^+(x) \leq R$  and  $D^-(x) \leq R$ . Then, in view of (1.4), we have

$$(2.2) \quad \begin{aligned} \text{dist}_{F+df}^+(p, x) &= D^+(x) + (f(x) - f(p)) \leq R + (f(x) - f(p)) \\ \text{dist}_{F+df}^-(p, x) &= D^-(x) + (f(p) - f(x)) \leq R + (f(p) - f(x)). \end{aligned}$$

Since  $\text{dist}_{F+df}^\pm(p, x) \geq 0$ , we obtain that  $-R \leq f(p) - f(x) \leq R$ . Then, the set  $B_R$  lies in the set  $\{x \in M \mid \text{dist}_{F+df}^+(p, x) + \text{dist}_{F+df}^-(p, x) \leq 3R\}$ , which, in its turn, lies in the set

$$(2.3) \quad \{x \in M \mid \text{dist}_{F+df}^+(p, x) \leq 3R\} \cap \{\text{dist}_{F+df}^-(p, x) \leq 3R\}.$$

Would the metric  $F + df$  be forward and backward complete, the set (2.3) would be compact implying all its closed subsets are compact which contradicts our assumption that  $B_R$  is not compact. Theorem is proved in one direction.

In order to prove it in the other direction, we consider the function

$$f : M \rightarrow \mathbb{R}, \quad f(x) := \frac{D^-(x) - D^+(x)}{2}.$$

The function  $f$  is not a priori smooth; next we show that the function is 1-Lipshitz w.r.t. the distance  $\text{dist}^+$ , that is, for every  $x, y \in M$  we have

$$(2.4) \quad \text{dist}^+(x, y) \geq f(x) - f(y).$$

Indeed, consider the triangles on the Fig. 1 and the corresponding triangle inequalities:

$$(2.5) \quad D^+(x) + \text{dist}^+(x, y) \geq D^+(y), \quad D^-(y) + \text{dist}^+(x, y) \geq D^-(x).$$

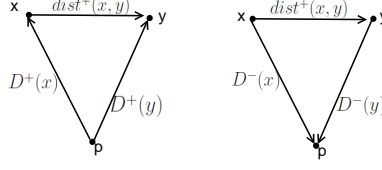


FIGURE 1. Triangles for the triangle inequalities.

The sum of these inequalities is equivalent to (2.4)

**Remark 2.** *As a consequence we obtain that the function  $f$  is also 1-Lipschitz w.r.t. the symmetrized distance  $dist^{sym} := dist^+ + dist^-$ . Since, locally, in a sufficiently small neighborhood, we can evidently find a euclidean structure such that the corresponding distance is not less than  $dist^{sym}$ , the function  $f$  is locally Lipschitz w.r.t. an Euclidean structure and is therefore differentiable at almost every point. Moreover, the restriction of the function to every smooth curve is a locally lipschitz function and the formula (1.3) remains valid though  $df$  is not everywhere defined.*

Let us now take a smooth function  $\tilde{f}$  on  $M$  such that

- (1)  $|\tilde{f}(x) - f(x)| \leq 1$  for all  $x \in M$ .
- (2)  $\tilde{f}(x)$  is 1.5-Lipschitz w.r.t.  $dist^+$ .

We will show that the existence of such a function in Appendix. Let us note that the proof of its existence essentially repeats the proof of the existence, for arbitrary  $\varepsilon_1, \varepsilon_2 > 0$ , of an  $\varepsilon_1$ -approximation of an 1-Lipschitz function by a smooth  $1 + \varepsilon_2$ -Lipschitz function on the standard  $\mathbb{R}^n$  with the standard metric.

Now let us take the function  $\frac{1}{2}\tilde{f}$  and consider  $F + \frac{1}{2}d\tilde{f}$ . This is a Finsler metric. Indeed, we need to check that  $F(x, v) + \frac{1}{2}d_x\tilde{f}(v) > 0$  for all  $x$  and for all  $v \neq 0$ . In a local coordinate system in a neighborhood of  $x$  we consider the curve  $t \mapsto x + t \cdot v$ ,  $t \in [0, \varepsilon]$ . From the definition (1.2) it follows that

$$(2.6) \quad F(x, v) = \lim_{t \rightarrow 0+} \frac{1}{t} dist^+(x, x + tv).$$

Now,

$$-d_x\tilde{f}(v) = \lim_{t \rightarrow 0+} \frac{1}{t}(\tilde{f}(x) - \tilde{f}(x + tv)) \stackrel{(2.4)}{\leq} \lim_{t \rightarrow 0+} \frac{1.5}{t} dist^+(x, x + tv) \stackrel{(2.6)}{=} 1.5F(x, v).$$

Then, since  $F(x, v) > 0$  for all  $v \neq 0$  we obtain  $-\frac{1}{2}d_x\tilde{f}(v) < F(x, v)$  for all  $v \neq 0$  implying  $F + \frac{1}{2}d\tilde{f}$  is a Finsler metric.

Let us now prove that the Finsler metric  $F + \frac{1}{2}d\tilde{f}$  is forward and backward complete. It is sufficient to show that for every  $r \in \mathbb{R}_{\geq 0}$  the balls

$$B_r^\pm(p) := \{x \in M \mid dist_{F + \frac{1}{2}d\tilde{f}}^\pm(p, x) \leq r\}$$

are compact. Indeed, any forward-Cauchy sequence lies in  $B_r^+(p)$  for sufficiently large  $r$ . Then, if such balls are compact, they are complete implying

our forward-Cauchy sequence converges. Similar arguments show that if all balls  $B_r^-(p)$  are compact then the metric is backward-complete.

Now, the  $(F + \frac{1}{2}d\tilde{f})$ -distance is given by

$$\begin{aligned} \text{dist}_{F+\frac{1}{2}d\tilde{f}}^+(p, x) &= \text{dist}_F^+(p, x) + \frac{1}{2}(\tilde{f}(x) - \tilde{f}(p)) \\ &\geq D^+(x) + \frac{1}{2}(f(x) - f(p)) - 1 \\ &= D^+(x) + \frac{1}{4}(D^-(x) - D^+(x)) - 1 \\ &= \frac{3}{4}D^+(x) + \frac{1}{4}D^-(x) - 1. \end{aligned}$$

As we explained in the beginning of the proof, since the function  $D^+ + D^-$  is proper, the function  $\frac{3}{4}D^+(x) + \frac{1}{4}D^-(x) - 1$  is proper as well implying the function  $\text{dist}_{F+\frac{1}{2}d\tilde{f}}^+(p, x)$  is proper implying the balls  $B_r(p)$  are compact so the metric  $F + \frac{1}{2}d\tilde{f}$  is forward-complete. The proof that the metric is backward-complete is similar. Theorem 1 is proved.

**Remark 3.** *In the proof we constructed a smooth function  $\tilde{f}$  such that the trivial projective change  $F + \frac{1}{2}\tilde{f}$  is a forward and backward complete Finsler metric. If we do not require the smoothness, we can simply take the trivial projective change corresponding to the function  $\frac{1}{2}f$ . As we have shown above, the function is locally Lipschitz so its differential is defined at almost every point so the function  $F + \frac{1}{2}df$  is defined almost everywhere. Moreover, for every curve  $c$  the formula (1.3) gives us a well-defined length (because the restriction of a locally Lipschitz function to a smooth curve is locally Lipschitz) and the length in  $F + \frac{1}{2}df$  is related to the length in  $F$  by the formula (1.4) so the (not everywhere defined) Finsler metric  $F + \frac{1}{2}df$  generates a forward- and backward complete distance function.*

### 3. APPENDIX: APPROXIMATING A LIPSCHITZ FUNCTION BY A SMOOTH FUNCTION

Let  $(M, F)$  be a Finsler manifold. Assume the function  $f$  is 1-Lipschitz w.r.t. to the distance function generated by  $F$ , that is for every  $x, y \in M$  we have

$$(3.1) \quad \text{dist}^+(x, y) \geq f(x) - f(y).$$

Our goal is to show that

for every  $\varepsilon_1, \varepsilon_2 > 0$  there exists a smooth function  $\tilde{f}$  such that

- $|\tilde{f}(x) - f(x)| < \varepsilon_1$  for all  $x$  and such that
- $\tilde{f}$  is  $(1 + \varepsilon_2)$ -Lipschitz w.r.t. to the distance function generated by  $F$ .

The special cases of this statement, when the  $F$  generates an euclidean distance or is a Riemannian metrics, are known: in the euclidean case, this is a well known folklore, and in the Riemannian case it was proved for example in [1].

Let us first do it in a small neighborhood of an arbitrary point  $p$ . We assume that the closure of the neighborhood is compact. We identify the neighborhood with a domain  $U' \subseteq \mathbb{R}^n$ , take a small number  $r > 0$  and a infinitely smooth positive function  $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}$  such that its support lies in the Euclidian  $r$ -ball, such that it is spherically symmetric with respect to 0 and such that the integral  $\int_{\mathbb{R}^n} \sigma(\xi) d\xi = 1$ . We denote by  $U$  the interior of the set of the points  $\{x \in U' \mid \bar{B}_r(x) \in U'\}$ . The set  $U$  is open and, if  $r$  is sufficiently small, contains the point  $p$ .

Now, denote by  $\tilde{f}_p$  the convolution of the function  $f$  with the function  $\sigma$ :

$$\tilde{f}_p(x) = \int_{\mathbb{R}^n} \sigma(x - \xi) f(\xi) d\xi.$$

The function  $\tilde{f}_p$  is defined for all  $x \in U$  and is smooth. Let us show that, if  $r$  is small enough,  $|\tilde{f}_p(x) - f(x)| \leq \varepsilon_1$  for all  $x$  and  $\tilde{f}_p$  is  $(1 + \varepsilon_2)$ -Lipschitz.

Since the function  $f$  is Lipschitz, it is uniformly continuous on  $U$  so for sufficiently small  $r$  we have  $|f(x) - f(y)| < \varepsilon_1$  for all  $x, y \in U$  such that  $d(x, y) < r$ . We consider

$$\begin{aligned} |\tilde{f}_p(x) - f(x)| &= \left| \int_{\mathbb{R}^n} \sigma(x - \xi) f(\xi) d\xi - \int_{\mathbb{R}^n} \sigma(x - \xi) f(x) d\xi \right| \\ &= \left| \int_{\mathbb{R}^n} \sigma(x - \xi) (f(\xi) - f(x)) d\xi \right| \\ &\leq \int_{\mathbb{R}^n} \sigma(x - \xi) \varepsilon_1 d\xi = \varepsilon_1. \end{aligned}$$

Let us show that, for a sufficiently small  $r$ , the function  $\tilde{f}_p$  is 1-Lipschitz. Since the function is smooth, it is sufficient to show that for every  $x$  and for every  $v$  the directional derivative of the function  $\tilde{f}_p$  at the point  $x$  in the direction  $v$  is less than  $(1 + \varepsilon_2)F(x, v)$ . Without loss of generality we can think that  $v = \frac{\partial}{\partial x^1}$ .

We have

$$\begin{aligned} \frac{\partial}{\partial x^1} \int_{\mathbb{R}^n} \sigma(x - \xi) f(\xi) d\xi &\stackrel{(1)}{=} \int_{\mathbb{R}^n} f(\xi) \frac{\partial}{\partial x^1} \sigma(x - \xi) d\xi \stackrel{(2)}{=} - \int_{\mathbb{R}^n} f(\xi) \frac{\partial}{\partial \xi^1} \sigma(x - \xi) d\xi \\ &\stackrel{(3)}{=} \int_{\mathbb{R}^n} \sigma(x - \xi) df \stackrel{(4)}{\leq} \int_{\mathbb{R}^n} \sigma(x - \xi) F\left(\xi, \frac{\partial}{\partial x^1}\right) d\xi. \end{aligned}$$

Let us explain the equalities/inequalities in the formula above.

- (1) Here we used the standard formula of the differentiation of an integral depending on the parameter (in this case,  $x^1$ ).
- (2) Here we used that  $\frac{\partial}{\partial x^1} \sigma(x - \xi) = -\frac{\partial}{\partial \xi^1} \sigma(x - \xi)$ .
- (3) Here we used integration by parts:  $\int u dv = uv - \int v du$ . The role of the function  $v$  plays the function  $\sigma(x - \xi)$ . The role of  $u$  plays the functions  $f(x)$  considered as a function of one variable  $x^1$ . Since  $f$  is Lipschitz, it has bounded variation so  $df$  is a well defined measure.



Now, for this choice of the functions  $u, v$ , the term  $uv|$  disappears since the function  $u = \sigma(x - \xi)$  has compact support, so we obtain  $\int u dv = -\int v du$  which gives us (3).

- (4) Here we used that  $\sigma$  is nonnegative and that the measure  $F(\xi, \frac{\partial}{\partial x^1}) d\xi$  is greater than  $df$ .

Now, since the Finsler function  $F$  is continuous, it is uniformly continuous on the unite spherical bundle  $S_1U$  implying that, if  $r$  is sufficiently small,  $F(\xi, \frac{\partial}{\partial x^1})$  is  $\varepsilon_2 \cdot F(x, \frac{\partial}{\partial x^1})$ -close to  $F(x, \frac{\partial}{\partial x^1})$  for  $\xi$  that are  $r$ -close to  $x$ . Then,

$$\int_{\mathbb{R}^n} \sigma(x-\xi) F(\xi, \frac{\partial}{\partial x^1}) d\xi \leq \int_{\mathbb{R}^n} \sigma(x-\xi) (1+\varepsilon_2) F(x, \frac{\partial}{\partial x^1}) d\xi = (1+\varepsilon_2) F(x, \frac{\partial}{\partial x^1})$$

implying that the  $v$ -derivative of the function  $\tilde{f}$  is not greater than  $(1 + \varepsilon_2)F(x, v)$  implying the function  $\tilde{f}$  is  $(1 + \varepsilon_2)$ -Lipschitz.

Thus, for every point  $p$  we can choose a neighborhood  $U_p$  such that for every  $\tilde{\varepsilon}_1 > 0, \tilde{\varepsilon}_2 > 0$  we can  $\tilde{\varepsilon}_1$ -approximate  $f$  in the neighborhood  $U_p$  by a  $(1 + \tilde{\varepsilon}_2)$ -Lipschitz function  $\tilde{f}_p$  on  $U_p$ . We take a locally finite cover  $U_p$ ,  $p \in P$  of  $M$  by such neighborhood and choose a smooth partition of unity  $\mu_p$  corresponding to this cover. We think that the approximations functions  $\tilde{f}_p$  are defined on the whole manifold (though it is not important what values do the functions  $\tilde{f}_p$  have on the points that do not lie in  $U_p$  since in all formulas below we will multiply  $\tilde{f}_p$  by  $\mu_p$  and all  $\mu_p$  are zero outside of  $U_p$ ).

Now, set

$$(3.2) \quad \tilde{f} := \sum_p \tilde{f}_p \cdot \mu_p.$$

The function  $\tilde{f}$  is well-defined since in a small neighborhood of every point  $x$  only finite many terms of the sum are not zero, and is evidently smooth. Let us show that we can chose the numbers  $\varepsilon_1(p), \varepsilon_2(p)$  for every  $U_p$  such that the function  $\tilde{f}$  satisfies our requirements.

Suppose a point  $x$  lies in the intersection of  $k$  neighborhoods of the cover  $U_p$ , which we denote by  $U_1, \dots, U_k$ . We will denote by  $\mu_1, \dots, \mu_k$  the corresponding elements of the partition of unity and by  $\tilde{f}_1, \dots, \tilde{f}_k$  the correspondent approximation  $\tilde{f}_p$ ; we will show that there exists  $\tilde{\varepsilon}_1, \tilde{\varepsilon}_2 > 0$  such that, if  $f_i$  are  $(1 + \tilde{\varepsilon}_1)$ -approximations of the restriction of the functions  $f$  to  $U_i$ , then  $\tilde{f}$  is a  $(1 + \varepsilon_1)$ -approximations of the restriction of the functions  $f$  to  $\bigcap_{i=1}^k U_i$ .

Indeed,

$$\tilde{f}(x) - f(x) = \mu_1(x)(\tilde{f}_1(x) - f(x)) + \dots + \mu_k(x)(\tilde{f}_k(x) - f(x)) \leq k \cdot \tilde{\varepsilon}_1.$$

Thus, for a  $\tilde{\varepsilon}_1 < \frac{1}{k}\varepsilon_1$  the function  $\tilde{f}$  is indeed an  $\varepsilon_1$ -approximation of  $f$ .

Let us now show that for a sufficiently small  $\tilde{\varepsilon}_1, \tilde{\varepsilon}_2 > 0$  the function  $\tilde{f}$  is indeed  $(1 + \varepsilon_2)$ -Lipschitz.

It is sufficient to prove that for every tangent vector  $v$  the directional derivative of  $\tilde{f}$  in the direction  $v$  is less than or equal to  $(1 + \varepsilon_2)F(x, v)$ . We take a point  $x$  such that it lies in the intersection of  $k$  elements of the cover which we again denote by  $U_1, \dots, U_k$ . Then  $\tilde{f}$  given by (3.2) is actually a finite sum

$$\tilde{f} = \mu_1 \tilde{f}_1 + \dots + \mu_k \tilde{f}_k.$$

Without loss of generality we can think that  $v = \frac{\partial}{\partial x^1}$ ; we need to show that

$$\frac{\partial}{\partial x^1}(\mu_1 \tilde{f}_1 + \dots + \mu_k \tilde{f}_k) \leq (1 + \varepsilon_2)F\left(x, \frac{\partial}{\partial x^1}\right).$$

The left hand side of the above inequality is equal, in view of the equalities  $\frac{\partial}{\partial x^1}\mu_1 + \dots + \frac{\partial}{\partial x^1}\mu_k = \frac{\partial}{\partial x^1}1 = 0$ , to

$$\begin{aligned} & \frac{\partial}{\partial x^1}(\mu_1 \tilde{f}_1 + \dots + \mu_k \tilde{f}_k) - \tilde{f} \cdot \left(\frac{\partial}{\partial x^1}\mu_1 + \dots + \frac{\partial}{\partial x^1}\mu_k\right) \\ &= (f_1 - \tilde{f}) \frac{\partial}{\partial x^1}\mu_1 + \dots + (f_k - \tilde{f}) \frac{\partial}{\partial x^1}\mu_k \\ &+ \underbrace{\mu_1 \frac{\partial}{\partial x^1}\tilde{f}_1 + \dots + \mu_k \frac{\partial}{\partial x^1}\tilde{f}_k}_{\leq (1+\varepsilon_2)F\left(x, \frac{\partial}{\partial x^1}\right)}. \end{aligned}$$

Now, since the functions  $\mu_i$  have bounded support, the derivatives  $\frac{\partial}{\partial x^1}\mu_i$  are bounded implying that the sum above is less than  $(1 + \varepsilon_2)F\left(x, \frac{\partial}{\partial x^1}\right)$  for sufficiently small  $\tilde{\varepsilon}_1, \tilde{\varepsilon}_2$  as we claimed.

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## REFERENCES

- [1] D. Azagra, J. Ferrera, F. Lopez-Mesas, Y. Rangel, *Smooth approximation of Lipschitz functions on Riemannian manifolds*, J. Math. Anal. Appl. **326**(2007) 1370–1378.
- [2] A. N. Bernal, M. Sánchez, *On smooth Cauchy hypersurfaces and Geroch's splitting theorem*, Comm. Math. Phys. **243**(2003), no. 3, 461–470.
- [3] A. N. Bernal, M. Sánchez, *Smoothness of time functions and the metric splitting of globally hyperbolic spacetimes*, Comm. Math. Phys. **257**(2005), no. 1, 43–50.
- [4] E. Caponio, M. A. Javaloyes, M. Sánchez, *On the interplay between Lorentzian Causality and Finsler metrics of Randers type*, Rev. Mat. Iberoamericana **27**(2011), no. 3, 919–952, arXiv:0903.3501
- [5] E. Caponio, M. A. Javaloyes, A. Masiello, *On the energy functional on Finsler manifolds and applications to stationary spacetimes*, Math. Ann., **351**(2011) 365–392, arXiv:math/0702323v4
- [6] E. Caponio, M.A. Javaloyes, A. Masiello, *Morse theory of causal geodesics in a stationary spacetime via Morse theory of geodesics of a Finsler metric*, Annales de l'Institut Henri Poincaré (C) Non Linear Analysis **27**(2010), 857–876, 2010.

- [7] E. Caponio, A. V. Germinario, M. Sánchez, *Geodesics on convex regions of stationary spacetimes and Finslerian Randers spaces*, arXiv:1112.3892
- [8] A. Dirmeier, M. Plaue, M. Scherfner, *Growth conditions, Riemannian completeness and Lorentzian causality*, to appear in J. Geom. Phys. (doi:10.1016/j.geomphys.2011.04.017)
- [9] M. A. Javaloyes, M. Sánchez, *A note on the existence of standard splittings for conformally stationary spacetimes*. Classical Quantum Gravity **25**(2008), no. 16, 168001.
- [10] I. Kovner, *Fermat principles for arbitrary space-times*, Astrophysical Journal, **351**(1990), pp. 114–120.